# III. COMPLEX NUMBERS

## A. Root of Negative One

Def.: imaginary unit i with $\boxed{\mathrm{i}^2 = -1}$ as symbolic solution to $x^2 = -1$
generalizes to complex numbers $\mathbb{C}$ : $z = a + \mathrm{i}b$, $a, b \in \mathbb{R}$
with $\mathrm{i}b$ understood as an (ordinary) product, hence $x^2 = -1$ implies $x = \pm\mathrm{i}$
$a = \mathsf{Re}(z)$: real part; $b = \mathsf{Im}(z)$: imaginary part $\longrightarrow$ $\mathbb{C} \sim \mathbb{R} \times \mathbb{R}$

complex numbers are indeed real objects (nothing imaginary out of this world)
we will observe that no further extension is needed to, *e.g.* solve $(x^2)^2 = -1$

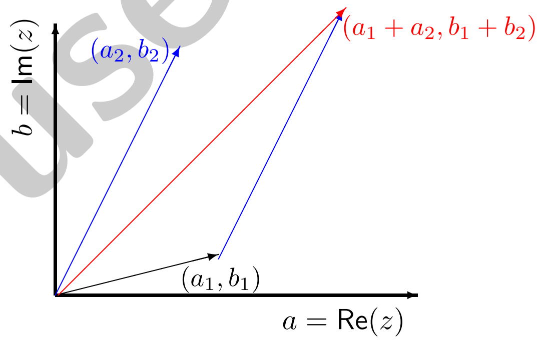
## B. Calculational Methods

addition of complex numbers, $z_1, z_2$

$$z_1 + z_2 = (a_1 + \mathrm{i}b_1) + (a_2 + \mathrm{i}b_2)$$
$$= (a_1 + a_2) + \mathrm{i}(b_1 + b_2) = a + \mathrm{i}b \in \mathbb{C}$$

just add real and imaginary parts separately

similar to vector addition when pictured in
a plane with real and imaginary parts




neutral element: $0$ : $a_2 = 0$, $b_2 = 0$ $\longrightarrow$ $z_1 + z_2 = z_1$
inverse element: $a_2 = -a_1$, $b_2 = -b_1$ $\longrightarrow$ $z_1 + z_2 = 0$ (neutral element)

multiplication of complex numbers, $z_1, z_2$
treat $\mathrm{i}b$ like the product $\mathrm{i} \cdot b$ with $\mathrm{i}^2 = -1$

$$z_1 z_2 = (a_1 + \mathrm{i}b_1)(a_2 + \mathrm{i}b_2) = a_1 a_2 + \mathrm{i}a_1 b_2 + \mathrm{i}b_1 a_2 + \mathrm{i}^2 b_1 b_2 = a_1 a_2 - b_1 b_2 + \mathrm{i}(a_1 b_2 + a_2 b_1)$$

$$\Longrightarrow \mathsf{Re}(z_1 z_2) = a_1 a_2 - b_1 b_2\,, \qquad \mathsf{Im}(z_1 z_2) = a_1 b_2 + a_2 b_1 \qquad \text{and} \qquad z_1 z_2 \in \mathbb{C}$$

in particular $\mathrm{i} \cdot \mathrm{i}$ has $a_{1,2} = 0$ and $b_{1,2} = 1$ so that $\mathsf{Re}(\mathrm{i} \cdot \mathrm{i}) = -1$ and $\mathsf{Im}(\mathrm{i} \cdot \mathrm{i}) = 0$ ✓
neutral element: $1$ : $a_2 = 1$, $b_2 = 0$ $\longrightarrow$ $z_1 z_2 = a_1 + \mathrm{i}b_1 = z_1$
inverse element: $z_1 z_2 = 1$ except for $z_1 = 0$ (distributivity requires $z \cdot 0 = 0 \neq 1$)
(recall distributivity: $z z_1 - z z_2 = z(z_1 - z_2)$ both must be zero when $z_1 = z_2$)
two conditions: $a_1 a_2 - b_1 b_2 = 1$ and $a_1 b_2 + a_2 b_1 = 0$, $a_1$ and $b_1$ are not both zero

(i) $a_1 \neq 0$ but $b_1$ can be any real number $\Longrightarrow$ $b_2 = -\frac{a_2}{a_1} b_1$
$\Longrightarrow$ $a_1 a_2 + \frac{a_2}{a_1} b_1^2 = 1$ $\Longrightarrow$ $a_2 = \frac{a_1}{a_1^2 + b_1^2}$ and $b_2 = \frac{-b_1}{a_1^2 + b_1^2}$

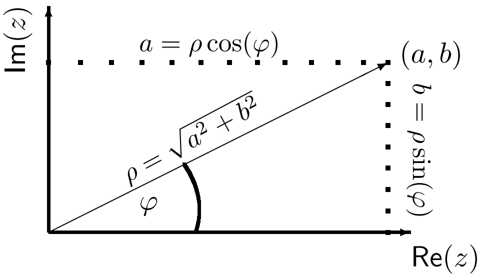
(ii) $b_1 \neq 0$ but $a_1$ can be any real number $\implies$ $a_2 = -\frac{b_2}{b_1} a_1$
$\implies$ $-a_1^2 \frac{b_2}{b_1} - b_1 b_2 = 1$ $\implies$ $b_2 = -\frac{b_1}{a_1^2+b_1^2}$ and $a_2 = \frac{a_1}{a_1^2+b_1^2}$

both cases:
$$(a_1 + \mathrm{i}b_1)\left(\frac{a_1}{a_1^2+b_1^2} - \mathrm{i}\frac{b_1}{a_1^2+b_1^2}\right) = 1$$

mathematical language: $\{\mathbb{C} = \mathbb{R} \times \mathbb{R}, " + ", " \cdot "\}$ is a *field*

**C. Polar Decomposition**

graphical representation of a complex number $z = a + \mathrm{i}b$ in a plane:




modulus (absolute value): $\rho = |z| = \sqrt{a^2 + b^2}$
phase: $\varphi$ with $a = \rho\cos(\varphi)$ and $b = \rho\sin(\varphi)$
$\varphi \in [0, 2\pi]$ or $[-\pi, \pi]$ (matter of convention)
$\varphi$ not defined for $z = 0$

naïvely: $\frac{b}{a} = \tan(\varphi)$ $\implies$ $\varphi = \arctan\left(\frac{b}{a}\right)$

but: $(i)$ $z = 1 + \mathrm{i}$ has $\dfrac{b}{a} = 1$ and $\varphi = \dfrac{\pi}{4}$
$(ii)$ $z = -1 - \mathrm{i}$ has $\dfrac{b}{a} = 1$ and $\varphi = \dfrac{\pi}{4}$ (should be $\frac{5\pi}{4}$ or $-\frac{3\pi}{4}$)

further specification: $b > 0 : \varphi \in [0, \pi]$ and $b < 0 : \varphi \in [\pi, 2\pi]$ or $[-\pi, 0]$
$b = 0 :$ $a > 0$ has $\varphi = 0$ while $a < 0$ has $\varphi = \pm\pi$

Def. <u>complex conjugation</u>: $z \longrightarrow z^*$ (or $\overline{z}$)
$a + \mathrm{i}b \longrightarrow a - \mathrm{i}b$ (swap sign of imaginary part)

consider
$$zz^* = (a + \mathrm{i}b)(a - \mathrm{i}b) = a^2 + b^2 = \rho^2 = |z|^2 \qquad \text{is real}$$

and
$$\frac{1}{z} = \frac{z^*}{zz^*} = \frac{z^*}{|z|^2} = \frac{a - \mathrm{i}b}{a^2 + b^2} \quad \checkmark \qquad \text{straightforward:} \quad (z_1 z_2)^* = z_1^* z_2^*$$

quadratic equation: $z^2 = -1 = \mathrm{i}^2$ $\implies$ $z_{1,2} = \pm\mathrm{i}$
generally: $z^2 + pz + qz = 0$ (take $p$ and $q$ real): $\left(z + \frac{p}{2}\right)^2 = \left(\frac{p}{2}\right)^2 - q$

(i) $\left(\frac{p}{2}\right)^2 \geq q$: proceed as in $\mathbb{R}$

(ii) $\left(\frac{p}{2}\right)^2 < q$: $\left(\frac{p}{2}\right)^2 - q = -\left|q - \left(\frac{p}{2}\right)^2\right| = \mathrm{i}^2\left|q - \left(\frac{p}{2}\right)^2\right|$ $\qquad z_{1,2} = -\frac{p}{2} \pm \mathrm{i}\sqrt{\left|q - \left(\frac{p}{2}\right)^2\right|}$

works as if $\mathrm{i} = \sqrt{-1}$ and $\sqrt{\left(\frac{p}{2}\right)^2 - q} = \sqrt{-1}\sqrt{\left|\left(\frac{p}{2}\right)^2 - q\right|}$
by accident because $p$-$q$ formula contains both signs

problems with the square root and complex numbers

- $a$ and $b$ real and positive: $\sqrt{ab} = \sqrt{a}\sqrt{b}$
  does not generalize: $1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = \mathrm{i}^2 = -1$ ?
- `mathematica`, for example: $\sqrt{-1+\mathrm{i}\epsilon} = \mathrm{i}$ while $\sqrt{-1-\mathrm{i}\epsilon} = -\mathrm{i}$ for $\epsilon \longrightarrow 0^+$
- $\sqrt[n]{x} = x^{\frac{1}{n}}$ related to $y = x^{\frac{1}{n}} \implies \ln(y) = \frac{1}{n}\ln(x)$
  need fractional power (or logarithm) of complex numbers

key to these questions: Euler formula for exponential function of imaginary number

$$\boxed{\mathrm{e}^{\mathrm{i}x} = \cos(x) + \mathrm{i}\sin(x)} \qquad x \in \mathbb{R}$$

(compare with $\mathrm{e}^{x} = \cosh(x) + \sinh(x)$)

derivation/motivation

- any complex number has a polar decomposition: $\mathrm{e}^{\mathrm{i}x} = r\,[\cos(\theta) + \mathrm{i}\sin(\theta)]$
- derivative of lhs (treat i just as a constant factor; *cf.* rules above)

$$\frac{d}{dx}\mathrm{e}^{\mathrm{i}x} = \mathrm{e}^{\mathrm{i}x}\frac{d}{dx}(\mathrm{i}x) = \mathrm{i}\mathrm{e}^{\mathrm{i}x} = r\,[\mathrm{i}\cos(\theta) - \sin(\theta)]$$

- derivative of rhs

$$\frac{dr}{dx}\,[\cos(\theta) + \mathrm{i}\sin(\theta)] + r\,[-\sin(\theta) + \mathrm{i}\cos(\theta)]\,\frac{d\theta}{dx}$$

- comparison: $\frac{dr}{dx} = 0$ and $\frac{d\theta}{dx} = 1 \implies r = c_1$ and $\theta = x + c_2$; $c_{1,2} \in \mathbb{R}$

$$x = 0: \qquad 1 = c_1\,[\cos(c_2) + \mathrm{i}\sin(c_2)] \qquad \implies \qquad c_1 = 1 \quad \text{and} \quad c_2 = 0 \qquad \checkmark$$

(equations for complex numbers relate to two real equations, one for real and imaginary parts each)

alternative derivation: Taylor series

reformulation of polar decomposition:

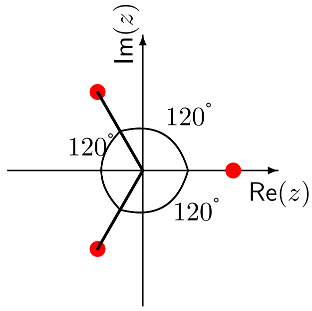
$$z = \rho\,[\cos(\varphi) + \mathrm{i}\sin(\varphi)] = \rho\mathrm{e}^{\mathrm{i}\varphi}$$

- logarithm: $\ln(z) = \ln(\rho) + \mathrm{i}\varphi$ is well defined complex number for $\rho > 0$
  (particular case $\rho = 0$ is singular just as it is for real numbers)
  logarithm is not unique: $\varphi$, $\varphi + 2\pi$, $\varphi + 4\pi, \ldots$ have same $z$ but different $\ln(z)$!
- roots and/or inverse powers: $z = \rho\mathrm{e}^{\mathrm{i}(\varphi+2\pi k)}$, $k = 0, 1, 2, \ldots$

$$z^{\frac{1}{n}} = \left[\rho\mathrm{e}^{\mathrm{i}(\varphi+2\pi k)}\right]^{\frac{1}{n}} = \sqrt[n]{\rho}\,\mathrm{e}^{\mathrm{i}\frac{\varphi}{n}}\,\mathrm{e}^{2\pi\mathrm{i}\frac{k}{n}}$$

different results for $k = 0, 1, \ldots, n-1 \implies n$ solutions to $z^n = c \quad (\neq 0)$
($k = n$ corresponds to $k = 0$, $k = n+1$ to $k = 1$, etc.)

example: $z^3 = 1\,, \quad z = 1$ is trivial




$$1 = \mathrm{e}^{\mathrm{i}0}\,,\ \mathrm{e}^{2\pi\mathrm{i}}\,,\ \mathrm{e}^{4\pi\mathrm{i}}$$

$$\longrightarrow \quad \mathrm{e}^{\mathrm{i}0/3}\,,\ \mathrm{e}^{2\pi\mathrm{i}/3}\,,\ \mathrm{e}^{4\pi\mathrm{i}/3}$$

$$= 1\,,\ -\frac{1}{2}\left(1-\sqrt{3}\,\mathrm{i}\right)\,,\ -\frac{1}{2}\left(1+\sqrt{3}\,\mathrm{i}\right)$$

$$\text{check}: \quad \left[-\frac{1}{2}\left(1-\sqrt{3}\,\mathrm{i}\right)\right]^3 = -\frac{1}{8}\left(1-\sqrt{3}\,\mathrm{i}\right)^2\left(1-\sqrt{3}\,\mathrm{i}\right) = -\frac{1}{8}\left(1-3-2\sqrt{3}\,\mathrm{i}\right)\left(1-\sqrt{3}\,\mathrm{i}\right)$$
$$= \frac{1}{4}\left(1+\sqrt{3}\,\mathrm{i}\right)\left(1-\sqrt{3}\,\mathrm{i}\right) = \frac{1}{4}(1+3) = 1$$

- root of $-1$ (convention with $\varphi \in [-\pi, \pi]$)

$$-1+\mathrm{i}\epsilon = 1\mathrm{e}^{\mathrm{i}\pi} \quad \Longrightarrow \quad \sqrt{-1+\mathrm{i}\epsilon} = \mathrm{e}^{\mathrm{i}\pi/2} = \mathrm{i}$$

$$-1-\mathrm{i}\epsilon = 1\mathrm{e}^{-\mathrm{i}\pi} \quad \Longrightarrow \quad \sqrt{-1+\mathrm{i}\epsilon} = \mathrm{e}^{-\mathrm{i}\pi/2} = -\mathrm{i}$$

discontinuity cannot be avoided:
for $\varphi \in [0, 2\pi]$ it occurs along the positive real axis with $\sqrt{1}$ jumping between $+1$ and $-1$

<u>addition theorem</u>

$$\mathrm{e}^{\mathrm{i}\varphi_1}\mathrm{e}^{\mathrm{i}\varphi_2} = [\cos(\varphi_1) + \mathrm{i}\sin(\varphi_1)]\,[\cos(\varphi_2) + \mathrm{i}\sin(\varphi_2)]$$
$$= \cos(\varphi_1)\cos(\varphi_2) - \sin(\varphi_1)\sin(\varphi_2) + \mathrm{i}\,[\sin(\varphi_1)\cos(\varphi_2) + \cos(\varphi_1)\sin(\varphi_2)]$$
$$= \cos(\varphi_1+\varphi_2) + \mathrm{i}\sin(\varphi_1+\varphi_2) = \mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)} \qquad \text{phases add in the product}$$

<u>analytic continuation,</u> examples with $x \in \mathbb{R}$

$$\cosh(\mathrm{i}x) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}\right) = \cos(x)$$
$$\sinh(\mathrm{i}x) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}\right) = \frac{1}{2}\left(\cos(x) + \mathrm{i}\sin(x) - \cos(x) + \mathrm{i}\sin(x)\right) = \mathrm{i}\sin(x)$$

## IV. INTRODUCTION TO ANALYTIC FUNCTION THEORY

Def.: a holomorphic function depends on one or more complex variables:

$$f = f(z_1, z_2, \ldots) \qquad (\text{not} \quad f(z_1, z_1^*, \ldots))$$

restricts combination on real and imaginary part of $z = x + \mathrm{i}y$, *e.g.* $z^2 = x^2 - y^2 + 2\mathrm{i}xy$
$f$ is complex differentiable in a domain $\subset \mathbb{C}^n$
for an *entire function* the domain is the entire complex plane

Def.: an analytic function is locally described by a converging power series
*e.g.* $f(z) = \sum_{i=0} c_i z^i$ for $|z| < z_0$, $z_0$ is the radius of convergence, may be infinite

separate real and imaginary parts

$$f(z) = u(x, y) + \mathrm{i}v(x, y)$$

though $f$ is a function of $z$, $u$ and $v$ are not separately functions of $z$
*e.g.* $f(z) = z^2 = x^2 - y^2 + 2\mathrm{i}xy \implies u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$

### A. Cauchy-Riemann relations

derivative of a holomorphic function: $f'(z) = \frac{df(z)}{dz} = \lim_{z_0 \to z} \frac{f(z) - f(z_0)}{z - z_0}$
approaching $z$ from different directions in the complex plane must produce identical results
recall real function: $g'(x) = \lim_{h \to 0^+} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0^-} \frac{g(x+h) - g(x)}{h}$

i) along the real axis: $z_0 = z + h$ with $h \in \mathbb{R}$

$$\begin{aligned} f'(z) &= \lim_{h \to 0} \frac{f(z) - f(z+h)}{-h} \\ &= \lim_{h \to 0} \frac{u(x, y) + \mathrm{i}v(x, y) - u(x+h, y) - \mathrm{i}v(x+h, y)}{-h} = u_x(x, y) + \mathrm{i}v_x(x, y) \end{aligned}$$

where $u_x(x, y) = \frac{\partial u(x,y)}{\partial x}$ etc.

ii) along the imaginary axis: $z_0 = z + \mathrm{i}h$ with $h \in \mathbb{R}$

$$\begin{aligned} f'(z) &= \lim_{h \to 0} \frac{f(z) - f(z+\mathrm{i}h)}{-\mathrm{i}h} \\ &= \lim_{h \to 0} \frac{u(x, y) + \mathrm{i}v(x, y) - u(x, y+h) - \mathrm{i}v(x, y+h)}{-\mathrm{i}h} = -\mathrm{i}u_y(x, y) + v_y(x, y) \end{aligned}$$

equating i) and ii) $\implies$ Cauchy-Riemann relations:

$$u_x(x, y) = v_y(x, y) \qquad \text{and} \qquad u_y(x, y) = -v_x(x, y)$$

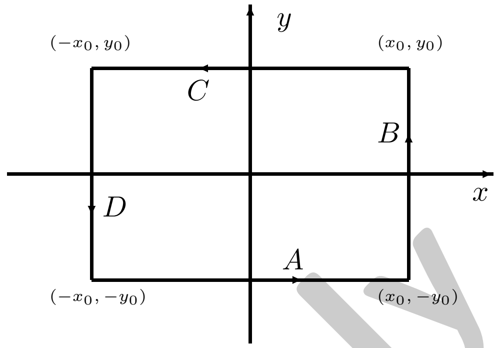
*e.g.* $f(z) = z^2 = x^2 - y^2 + 2\mathrm{i}xy$ has $u = x^2 - y^2$ and $v = 2xy$
has $u_x = 2x\,, v_y = 2x\,, u_y = -2y\,, v_x = 2y$ ✓

interesting consequence

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u_x}{\partial x} = \frac{\partial v_y}{\partial x} = \frac{\partial v_x}{\partial y} = -\frac{\partial u_y}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$$

real part of an analytic function solves Laplace's equation in two dimensions $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$
so does the imaginary part

## B. Residue theorem




closed counter-clockwise integral in the complex plane

$x_0$ and $y_0$ are arbitrarily small

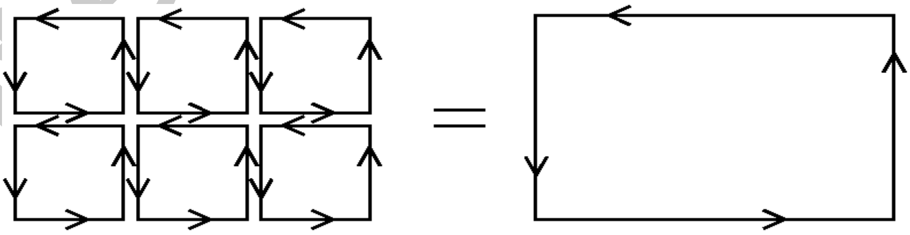
$$
\begin{aligned}
\oint_{\mathcal{C}} dz\, f(z) &= \oint_{\mathcal{C}} [dx + \mathrm{i}dy]\,[u(x,y) + \mathrm{i}v(x,y)] \\
&= \int_{-x_0}^{x_0} dx \left[ \underbrace{u(x,-y_0) + \mathrm{i}v(x,-y_0)}_{A} \underbrace{-u(x,y_0) - \mathrm{i}v(x,y_0)}_{C} \right] \\
&\quad + \mathrm{i} \int_{-y_0}^{y_0} dy \left[ \underbrace{u(x_0,y) + \mathrm{i}v(x_0,y)}_{B} \underbrace{-u(-x_0,y) - \mathrm{i}v(-x_0,y)}_{D} \right] \\
&\approx \int_{-x_0}^{x_0} dx\, 2y_0\, [-u_y(x,0) - \mathrm{i}v_y(x,0)] + \mathrm{i} \int_{-y_0}^{y_0} dy\, 2x_0\, [u_x(0,y) + \mathrm{i}v_x(0,y)] \\
&\approx 4x_0y_0\, [-u_y(\widetilde{x},0) - \mathrm{i}v_y(\widetilde{x},0) + \mathrm{i}u_x(0,\widetilde{y}) - v_x(0,\widetilde{y})]
\end{aligned}
$$

with $-x_0 < \widetilde{x} < x_0$ and $-y_0 < \widetilde{y} < y_0$ by the central value theorem of integration
take the infinitesimal limit $x_0 \to 0$ and $y_0 \to 0$, *i.e.* $\widetilde{x} \to 0$ and $\widetilde{y} \to 0$:

$$
\oint_{\mathcal{C}} dz\, f(z) \longrightarrow 4x_0y_0\, [-u_y(0,0) - \mathrm{i}v_y(0,0) + \mathrm{i}u_x(0,0) - v_x(0,0)] = 0
$$

by the Cauchy-Riemann relations[8]

patch finite area by (infinitely) many infinitesimal loops



contributions from internal lines to $\oint_{\mathcal{C}} dz\, f(z)$ cancel
conclusion: $\oint_{\mathcal{C}} dz\, f(z) = 0$ if the central value theorem can be applied

what happens for a singular structure like $f(z) = \frac{a}{z - z_0}$ $\big(a, z_0 \in \mathbb{C}\big)$?
notation: $z_0$ is the <u>pole</u> (position) and $a$ is the <u>residue</u>
let $\mathcal{C} = \partial\mathcal{A}$ (boundary of an area)

[8] Calculation essentially is Stokes's theorem with $\boldsymbol{dr} = dx\boldsymbol{e}_x + dy\boldsymbol{e}_y$ and $\boldsymbol{V} = [u(x,y) + \mathrm{i}v(x,y)]\,[\boldsymbol{e}_x + \mathrm{i}\boldsymbol{e}_y]$.

i) $z_0 \notin \mathcal{A} \quad \Longrightarrow \quad$ central value theorem applies $\quad \Longrightarrow \quad \oint_{\mathcal{C}} dz\, f(z) = 0$

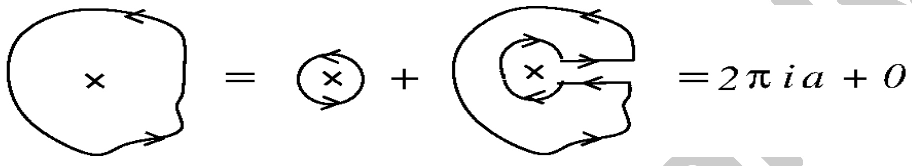
ii) $z_0 \in \mathcal{A}$ counter-clockwise circle (fixed radius $R$) around pole

$$\mathcal{C}: \quad z = z_0 + R\mathrm{e}^{\mathrm{i}\varphi} \qquad 0 \le \varphi < 2\pi$$

$$dz = \mathrm{i}R\mathrm{e}^{\mathrm{i}\varphi}\, d\varphi \qquad \Longrightarrow \qquad \oint_{\mathcal{C}} dz\, f(z) = \mathrm{i}Ra \int_0^{2\pi} d\varphi\, \frac{\mathrm{e}^{\mathrm{i}\varphi}}{R\mathrm{e}^{\mathrm{i}\varphi}} = 2\pi\mathrm{i}a$$

independent of radius

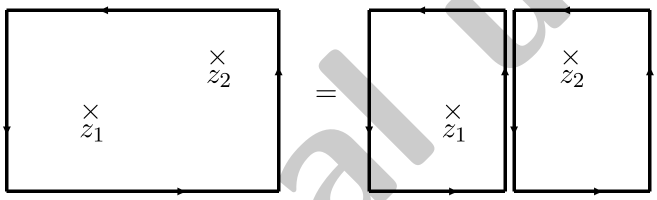
separate pole for an arbitrary contour




gap only shown for illustration

a closed counter-clockwise contour integral that encircles a (first order) pole yields $2\pi\mathrm{i}\times$(residue of that pole); clockwise contour has the negative thereof

iii) multiple poles: $f(z) = \frac{a_1}{z-z_1} + \frac{a_2}{z-z_2}\,, \quad z_1 \neq z_2$
patch contours encircling individual poles




$$\oint_{\mathcal{C}} dz\, f(z) = 2\pi\mathrm{i} \sum_i a_i$$

higher order poles: $f(z) = \frac{a}{(z-z_0)^n}$, with integer $n \ge 2$; obviously $\oint_{\mathcal{C}} dz\, f(z) = 0$ when $z_0 \notin \mathcal{A}$

as above: $\mathcal{C}: \quad z = z_0 + R\mathrm{e}^{\mathrm{i}\varphi}$

$$\oint_{\mathcal{C}} dz\, f(z) = \mathrm{i}aR \int_0^{2\pi} d\varphi \frac{\mathrm{e}^{\mathrm{i}\varphi}}{R^n \mathrm{e}^{\mathrm{i}n\varphi}} = \mathrm{i}aR^{1-n} \int_0^{2\pi} d\varphi\, \mathrm{e}^{\mathrm{i}(n-1)\varphi} = 0 \qquad \text{since} \quad n \neq 1$$

final result for counter-clockwise contour integral:

$$\oint_{\mathcal{C}} dz\, f(z) = 2\pi\mathrm{i} \times \big(\text{sum of residues of first order poles encircled by } \mathcal{C}\big)$$

## Calculational methods

i) $f(z)$ has a first order pole at $z = z_0$

$$a = \mathrm{Res}_{z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z)$$

ii) $f(z)$ has an $m^{\mathrm{th}}$ order pole at $z = z_0$

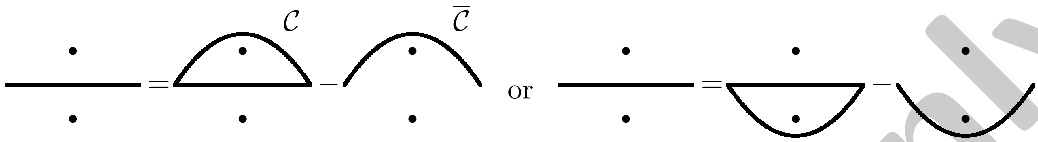
$$\mathrm{Res}_{z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z)$$

### C. Applications

extend ordinary integrals (along real axis) to closed contour integrals
& ensure that extension contributes zero

Examples

A) $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \int_{-\infty}^{\infty} \frac{dx}{(x+\mathrm{i})(x-\mathrm{i})}$ has first order poles at $z_0 = \pm\mathrm{i}$




extension of integral $\overline{\mathcal{C}}$ : $z = R\mathrm{e}^{\mathrm{i}\varphi}$ with $0 \leq \varphi \leq \pi$

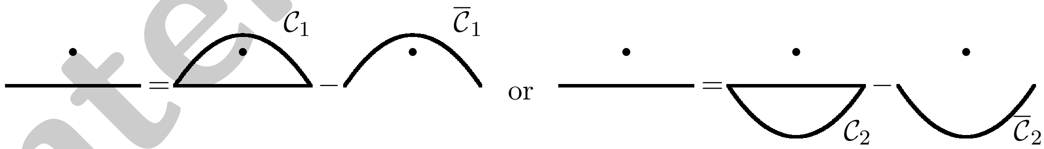
$$\int_{\overline{\mathcal{C}}} \frac{dz}{z^2+1} = \frac{\mathrm{i}}{R} \int_0^{\pi} \frac{d\varphi}{\mathrm{e}^{2\mathrm{i}\varphi} + 1/R^2} \longrightarrow 0 \quad \text{when} \quad R \longrightarrow \infty$$

extension does not contribute $\Longrightarrow$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \int_{\mathcal{C}} \frac{dz}{(z+\mathrm{i})(z-\mathrm{i})} = 2\pi\mathrm{i}\mathrm{Res}_{\mathrm{i}} \frac{1}{(z+\mathrm{i})(z-\mathrm{i})} = 2\pi\mathrm{i} \lim_{z\to\mathrm{i}} \frac{z-\mathrm{i}}{(z+\mathrm{i})(z-\mathrm{i})} = \frac{2\pi\mathrm{i}}{2\mathrm{i}} = \pi$$

alternative extension has residue with opposite sign
but an additional factor (-1) from the contour being clockwise

B) $\theta(s) = \int_{-\infty}^{\infty} \frac{dx}{2\pi\mathrm{i}} \frac{\mathrm{e}^{\mathrm{i}sx}}{x-\mathrm{i}\epsilon}$ has a first order pole at $z = \mathrm{i}\epsilon$
indication that the pole is an infinitesimal amount above the real axis: $\epsilon = 0^+$
alternatively: pole is bypassed in the lower half-plane
it is permissable to multiply $\epsilon$ by any positive real number (sign unchanged)




semi-circle extensions contribute zero when the exponential vanishes
on the semi-circles: $z = R\mathrm{e}^{\mathrm{i}\varphi}$ so that $\mathrm{e}^{\mathrm{i}sz} = \mathrm{e}^{\mathrm{i}sR\cos\varphi}\,\mathrm{e}^{-sR\sin\varphi}$
vanishes for $R \to \infty$ when $s\sin\varphi > 0$, $\overline{\mathcal{C}}_1 :\ \sin\varphi > 0$, $\overline{\mathcal{C}}_2 :\ \sin\varphi < 0$
for $s > 0$ we need to take $\mathcal{C}_1$

$$\theta(s) = \mathrm{Res}_{\mathrm{i}\epsilon} \frac{\mathrm{e}^{\mathrm{i}sx}}{x-\mathrm{i}\epsilon} = \mathrm{e}^{-s\epsilon} \longrightarrow 1 \qquad \text{because} \quad \epsilon \to 0^+$$

for $s < 0$ we need to take $\mathcal{C}_2$ which has $\theta(s) = 0$
result: $\int_{-\infty}^{\infty} \frac{dx}{2\pi\mathrm{i}} \frac{\mathrm{e}^{\mathrm{i}sx}}{x-\mathrm{i}\epsilon}$ is a parameterization of the step function

derivative of step function (assuming it can be pulled under the integral)

$$\frac{d\theta(s)}{ds} = \int \frac{dx}{2\pi} \frac{x}{x - \mathrm{i}\epsilon} \mathrm{e}^{\mathrm{i}sx} = \frac{1}{\mathrm{i}} \begin{cases} \mathrm{Res}_{\mathrm{i}\epsilon} \dfrac{x}{x-\mathrm{i}\epsilon} \mathrm{e}^{\mathrm{i}sx} = \mathrm{i}\epsilon \mathrm{e}^{-s\epsilon} = 0 & s > 0 \\ \text{undetermined} & s = 0 \\ 0 & s < 0 \end{cases}$$

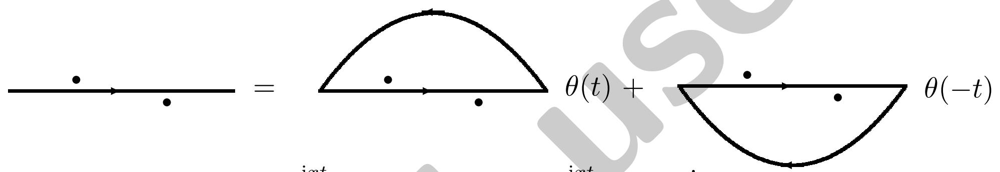
test function $g(s)$ and integration bounds $a < 0 < b$

$$\int_a^b ds\, g(s) \frac{d\theta(s)}{ds} = [g(s)\theta(s)]_a^b - \int_a^b ds\, \frac{dg(s)}{ds} \theta(s) = g(b) - \int_0^b ds\, \frac{dg(s)}{ds} = g(0)$$

hence $\frac{d\theta(s)}{ds} = \delta(s)$, the Dirac-$\delta$ *function* (distribution)
since the residue above is zero, we may omit the i$\epsilon$ prescription and write

$$\delta(s) = \int \frac{dx}{2\pi} \mathrm{e}^{\mathrm{i}sx} \qquad \text{(consistent with Fourier transform:} \quad g(x) = \int \tfrac{dk}{2\pi} \int dx'\, g(x') \mathrm{e}^{\mathrm{i}(x-x')k}\text{)}$$

C) $S(t) = \int \frac{dx}{2\pi} \frac{\mathrm{e}^{\mathrm{i}xt}}{x^2 - m^2 + \mathrm{i}\epsilon}$ has first order poles at $x = \pm\sqrt{m^2 - \mathrm{i}\epsilon} \approx \pm\left(m - \frac{\mathrm{i}\epsilon}{2m}\right) \to \pm m \mp \mathrm{i}\epsilon$




$$S(t) = \mathrm{i}\theta(t) \mathrm{Res}_{-m} \frac{\mathrm{e}^{\mathrm{i}xt}}{x^2 - m^2} - \mathrm{i}\theta(-t) \mathrm{Res}_m \frac{\mathrm{e}^{\mathrm{i}xt}}{x^2 - m^2} = \frac{-\mathrm{i}}{2m} \left[\theta(t)\mathrm{e}^{-\mathrm{i}mt} + \theta(-t)\mathrm{e}^{\mathrm{i}mt}\right] = \frac{-\mathrm{i}}{2m} \mathrm{e}^{-\mathrm{i}m|t|}$$

consider

$$\left(-\frac{d^2}{dt^2} - m^2\right) S(t) = \frac{1}{2} \frac{d}{dt} \left[\mathrm{e}^{-\mathrm{i}m|t|} \mathrm{sgn}(t)\right] + \frac{\mathrm{i}m}{2} \mathrm{e}^{-\mathrm{i}m|t|} = \frac{1}{2} \mathrm{e}^{-\mathrm{i}m|t|} \frac{d}{dt} \left[\theta(t) - \theta(-t)\right] = \delta(t)$$

on the other hand

$$\left(-\frac{d^2}{dt^2} - m^2\right) S(t) \int \frac{dx}{2\pi} \frac{\mathrm{e}^{\mathrm{i}xt}}{x^2 - m^2 + \mathrm{i}\epsilon} = \int \frac{dx}{2\pi} \frac{x^2 - m^2}{x^2 - m^2 + \mathrm{i}\epsilon} \mathrm{e}^{\mathrm{i}xt} \quad \longrightarrow \quad \int \frac{dx}{2\pi} \mathrm{e}^{\mathrm{i}xt} = \delta(t)$$

physics application: time-ordered product, Feynman propagator

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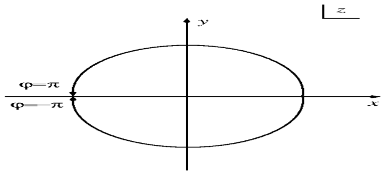
branch cut singularities ($\epsilon = 0^+, \quad x > 0$)

$\ln(-x + \mathrm{i}\epsilon) - \ln(-x - \mathrm{i}\epsilon) = 2\pi\mathrm{i} \ln(x)$

$\sqrt{z} = \mathrm{e}^{\ln\left(\sqrt{z}\right)} = \mathrm{e}^{\frac{1}{2}\ln(z)}$

$\sqrt{-x + \mathrm{i}\epsilon} - \sqrt{-x - \mathrm{i}\epsilon} = \sqrt{x}\left(\sqrt{\mathrm{e}^{\mathrm{i}\pi}} - \sqrt{\mathrm{e}^{-\mathrm{i}\pi}}\right)$

$= \sqrt{x}\left(\mathrm{e}^{\mathrm{i}\pi/2} - \mathrm{e}^{-\mathrm{i}\pi/2}\right) = \sqrt{x}\,(\mathrm{i} - (-\mathrm{i})) = 2\pi\mathrm{i}\sqrt{x}$




can be used to express an integral along real axis as an integral along the imaginary axis

*e.g.*: $\int_0^\infty dx \frac{\cos(x)}{\sqrt{x^2+1}} = \frac{1}{2}\int_{-\infty}^\infty dx \frac{\mathrm{e}^{\mathrm{i}x}}{\sqrt{x^2+1}} = \int_1^\infty dx \frac{\mathrm{e}^{-x}}{\sqrt{x^2-1}}$